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Asymptotic Properties of Orthogonal Polynomials from Their Recurrence Formula, I

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In this paper it is supposed that the coefficients in the recurrence formula for orthogonal polynomials have finite limits as the index goes to infinity over the set of even and odd integers. The asymptotic behavior of the ratio of two contiguous polynomials and the limiting zero distribution are discussed. Applications to quadrature formulas are given. © 1985 Academic Press, Inc.

I. INTRODUCTION

Let $\{q_n(x)\}$ be a sequence of monic polynomials that satisfy a recurrence relation of the form

$$\begin{aligned} q_{n+1}(x) &= (x-\alpha_n)\, q_n(x) - \beta_n q_{n-1}(x); \qquad n = 0, 1, 2, \ldots, \\ q_{-1}(x) &= 0; \quad q_0(x) = 1; \quad \alpha_n \in \mathbb{R}, \quad \beta_n > 0. \end{aligned} \tag{1.1}$$

It is well known that there exists a distribution function $W(x)$ on the real line such that the polynomials $\{q_n(x)\}$ satisfy the orthogonality relations

$$\int_{-\infty}^{+\infty} q_n(x)\, q_m(x)\, dW(x) = \gamma_n^{-2}\, \delta_{m,n}. \tag{1.2}$$

On the other hand, orthonormal polynomials always satisfy a relation of the form (1.1), with $\beta_n = (\gamma_{n-1}/\gamma_n)^2$ [9, Theorems I.2.1 and II.1.5]. The polynomial $q_n(x)$ has n real and distinct zeros, $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$, and these zeros belong to the interval on which $W(x)$ is concentrated [9, Theorem I.2.2].

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In this paper we assume that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in (1.1) have the following asymptotic behavior

$$\lim_{n\to\infty} \alpha_{2n} = a_1; \qquad \lim_{n\to\infty} \beta_{2n} = b_1^2;$$
$$\lim_{n\to\infty} \alpha_{2n+1} = a_2; \qquad \lim_{n\to\infty} \beta_{2n+1} = b_2^2. \tag{1.3}$$

If we use the estimate

$$|x_{k,n}| \leqslant \max_{0\leqslant j\leqslant n-1} |\alpha_j| + 2 \max_{1\leqslant j\leqslant n-1} \beta_j^{1/2}$$

[15, Formula (11)], then we can conclude that the zeros of the polynomials $\{q_n(x)\}$ are always in a compact interval, say $[-A, A]$.

Nevai [13, 14] made a thorough investigation of the case $a_1 = a_2 = a$ and $b_1 = b_2 = b$ (in [14] b was supposed to be greater than zero). One of his results is that the zeros of the corresponding orthogonal polynomials have regular arcsine behavior. This behavior of the zeros was already known for a great class of orthogonal polynomials (see, e.g., [8] and [19]), but in most cases the result follows from a priori knowledge of the distribution function W with respect to which the polynomials are orthogonal. However, one may only have access to the recurrence formula (1.1) without any knowledge of the distribution function W. Chihara [6, 7] shows that under (1.3) (with $b_1 = b_2 = b$) one can use chain sequences to prove that the zeros of the orthogonal polynomials are dense on the union of two disjoint intervals. These chain sequences were also used by Maki [12] to prove the regular arcsine behavior for the case $a_1 = a_2 = a$ and $b_1 = b_2 = b$, but he made the extra assumption that $\beta_n \leqslant b^2$.

Akhiezer [1] studied orthogonal polynomials with a weight function that is concentrated on the union of two disjoint intervals and these polynomials have recurrence coefficients that satisfy (1.3). These two disjoint intervals are typical for the kind of asymptotic behavior of the recurrence coefficients that we will investigate, this was already apparent from Chihara's result mentioned earlier. The case where the two intervals are reduced to two points needs special attention: Krein [2, Article VI] gave necessary and sufficient conditions on the recurrence coefficients for the only limit points of the support of W to be a finite number of given points. In Section IV we will show that these cases appear when b_1 or b_2 is equal to zero.

In this paper we will prove some asymptotic properties of the polynomials $\{q_n(x)\}$ under the condition (1.3), in particular we will give the asymptotic value of the ratio of two orthogonal polynomials the index of which differ by one or two units. We will use these asymptotics to obtain the zero distribution of these polynomials. As an application we will give some quadrature formulas associated to these asymptotics. Let us note that Geronimus [11] also obtained the zero distribution for these polynomials by using potential theory.

II. Asymptotic Properties

From the recurrence formula (1.1) we easily obtain

$$q_{2n+1}(x) = (x-\alpha_{2n})\, q_{2n}(x) - \beta_{2n} q_{2n-1}(x),$$

$$q_{2n+2}(x) = (x-\alpha_{2n+1})\, q_{2n+1}(x) - \beta_{2n+1} q_{2n}(x).$$

Solving the second equation for $q_{2n+1}(x)$ yields

$$q_{2n+1}(x) = \frac{q_{2n+2}(x) + \beta_{2n+1} q_{2n}(x)}{x-\alpha_{2n+1}},$$

and putting this in the first equation gives the following recurrence formula for the even-indexed polynomials

$$q_{2n+2}(x) = \left[(x-\alpha_{2n})(x-\alpha_{2n+1}) - \beta_{2n+1} - \beta_{2n} \frac{x-\alpha_{2n+1}}{x-\alpha_{2n-1}} \right] q_{2n}(x) - \beta_{2n}\beta_{2n-1} \frac{x-\alpha_{2n+1}}{x-\alpha_{2n-1}}\, q_{2n-2}(x), \tag{2.1}$$

and a similar reasoning leads to

$$q_{2n+3}(x) = \left[(x-\alpha_{2n+1})(x-\alpha_{2n+2}) - \beta_{2n+2} - \beta_{2n+1} \frac{x-\alpha_{2n+2}}{x-\alpha_{2n}} \right] q_{2n+1}(x) - \beta_{2n+1}\beta_{2n} \frac{x-\alpha_{2n+2}}{x-\alpha_{2n}}\, q_{2n-1}(x). \tag{2.2}$$

We will use these modified recurrence formulas to prove asymptotics for the polynomials $q_n(x)$ as n tends to infinity. We need some notation: let X_1 be the set of accumulation points of the set $\{x_{i,n} \mid i = 1,..., n;\ n = 1, 2,...\}$ and $X_2 = \{x \in \mathbb{R} \mid q_n(x) = 0$ for infinitely many $n\}$. An element of X_2 is not necessarily an accumulation point of the set $\{x_{i,n}\}$: if we take a weight function on $[-\beta, -\alpha] \cup [\alpha, \beta]$ $(0 < \alpha < \beta)$ that is symmetric around the origin, then the origin will be in X_2 since every orthogonal polynomial of odd degree will be zero at the origin, but we can not find a sequence of zeros (except for the constant zero sequence) that converges to zero. Indeed, polynomials of even degree will not vanish inside $(-\alpha, \alpha)$ because if there is one zero in that interval then by symmetry there will also be a second zero which is impossible (orthogonal polynomials can have at most one zero in an interval where the distribution function W is constant). By the same reasoning the origin will be the only zero in $(-\alpha, \alpha)$ for the polynomials of odd degree.

It is well known that $S(W) \subset X_1 \cup X_2$, where $S(W) = \{x \in \mathbb{R} \mid W(x+\varepsilon) - W(x-\varepsilon) > 0 \text{ for all } \varepsilon > 0\}$ is the spectrum of the distribution function W [7, p. 60], and the spectrum may be different from $X_1 \cup X_2$, as can be seen from the previous remark. By the notation $f_n(x) \sim g(x)$ we mean that the ratio $f_n(x)/g(x)$ tends to one. The Riemann sphere $\mathbb{C} \cup \{\infty\}$ is denoted by $\bar{\mathbb{C}}$. Finally we define $Z_N = \{x_{i,n} \mid i = 1,..., n;\ n \geqslant N\}$.

THEOREM 1. *If the coefficients in the recurrence formula* (1.1) *satisfy* (1.3) *then as* $n \to \infty$

$$\frac{q_n(z)}{q_{n-2}(z)} \sim Q(z) = \tfrac{1}{2}\{(z-a_1)(z-a_2) - (b_1^2 + b_2^2) + \sqrt{[(z-a_1)(z-a_2) - (b_1^2+b_2^2)]^2 - 4b_1^2 b_2^2}\} \tag{2.3}$$

uniformly on compact subsets of $\bar{\mathbb{C}} \backslash (X_1 \cup X_2)$.

Proof. In the Introduction we indicated that the zeros of $\{q_n(x)\}$ are all in a compact interval $[-A, A]$, hence the ratio $q_n(z)/q_{n-2}(z)$ is analytic in $\mathbb{C} \backslash [-A, A]$ for every $n \geqslant 2$. If K is a compact set in $\bar{\mathbb{C}} \backslash (X_1 \cup X_2)$, then K can have at most a finite number of zeros of $\{q_n(x)\}$ and each of these are only a finite number of times a zero. This means that there exists an integer N such that for $n \geqslant N$ the ratios $q_n(z)/q_{n-2}(z)$ are analytic in K and for $z \in K$ we have for $n \geqslant N$,

$$\left|\frac{q_{n-2}(z)}{q_n(z)}\right| = \left|\frac{q_{n-2}(z)}{q_{n-1}(z)}\right| \left|\frac{q_{n-1}(z)}{q_n(z)}\right| \leqslant \sum_{j=1}^{n-1} \frac{a_{j,n-1}}{|z - x_{j,n-1}|} \sum_{k=1}^{n} \frac{a_{k,n}}{|z - x_{k,n}|}$$
$$\leqslant \frac{1}{\delta^2} \sum_{j=1}^{n-1} a_{j,n-1} \sum_{k=1}^{n} a_{k,n},$$

here we used a decomposition in partial fraction, and we have put $\delta = \inf \{|z-x| \mid z \in K,\ x \in (X_1 \cup X_2) \cap Z_N\}$ which is a strictly positive quantity because K is a compact set and $(X_1 \cup X_2) \cap Z_N$ is a closed set, while K and $(X_1 \cup X_2) \cap Z_N$ are disjoint. Note that [18, p. 47]

$$a_{j,n} = \frac{q_{n-1}(x_{j,n})}{q_n'(x_{j,n})} > 0,$$

$$\sum_{j=1}^{n} a_{j,n} = \lim_{z \to \infty} \frac{z q_{n-1}(z)}{q_n(z)} = 1.$$

Hence the ratio $q_{n-2}(z)/q_n(z)$ is uniformly bounded on every compact subset of $\bar{\mathbb{C}}\backslash(X_1 \cup X_2)$. Next we will prove that this ratio converges when $z \in [A, \infty)$, and since this set has a limit point, we can use the theorem of *Stieltjes–Vitali* [7, p. 121] to conclude the uniform convergence on compact subsets of $\bar{\mathbb{C}}\backslash(X_1 \cup X_2)$.

If we take $z \in [A, \infty)$, then the coefficients in the recurrence formulas (2.1) and (2.2) converge as $n \to \infty$. We may therefore use a famous result of Poincaré [16] to conclude that both $q_{2n+2}(z)/q_{2n}(z)$ and $q_{2n+1}(z)/q_{2n-1}(z)$ converge as $n \to \infty$. The convergence of these ratios also follows from the fact that the sequence

$$
\begin{aligned}
f_n(z) = \beta_{2n}\beta_{2n-1}\,&\frac{z-\alpha_{2n+1}}{z-\alpha_{2n-1}} \\
&\times \left\{(z-\alpha_{2n})(z-\alpha_{2n+1}) - \beta_{2n+1} - \beta_{2n}\frac{z-\alpha_{2n+1}}{z-\alpha_{2n-1}}\right\}^{-1} \\
&\times \left\{(z-\alpha_{2n-2})(z-\alpha_{2n-1}) - \beta_{2n-1} - \beta_{2n-2}\frac{z-\alpha_{2n-1}}{z-\alpha_{2n-3}}\right\}^{-1}
\end{aligned}
$$

is a chain sequence with (minimal) parameter sequence

$$
g_n(z) = 1 - \frac{q_{2n+2}(z)}{q_{2n}(z)}\left\{(z-\alpha_{2n})(z-\alpha_{2n+1}) - \beta_{2n+1} - \beta_{2n}\frac{z-\alpha_{2n+1}}{z-\alpha_{2n-1}}\right\}^{-1}
$$

(this means $f_n(z) = g_n(z)[1 - g_{n-1}(z)]$), and since $f_n(z)$ converges for $z \in [A, \infty)$ also $g_n(z)$ will converge [7, p. 102]. To determine the limit we divide Eq. (2.1) by $q_{2n}(z)$ and Eq. (2.2) by $q_{2n+1}(z)$. Then let $n \to \infty$ to obtain

$$
Q(z) = \{(z-a_1)(z-a_2) - (b_1^2 + b_2^2)\} - \frac{b_1^2 b_2^2}{Q(z)}.
$$

If we solve this equation we get

$$
\begin{aligned}
Q(z) = \tfrac{1}{2}\{&(z-a_1)(z-a_2) - (b_1^2+b_2^2) \\
&\pm \sqrt{[(z-a_1)(z-a_2) - (b_1^2+b_2^2)]^2 - 4b_1^2 b_2^2}\}.
\end{aligned}
$$

Since $q_n(z)/q_{n-2}(z) \to \infty$ as $z \to \infty$ we have to choose the positive sign in this solution, so that also $Q(z) \to \infty$ as $z \to \infty$. ∎

The asymptotic relation does not hold on $X_1 \cup X_2$ since on this set the ratio $q_n(z)/q_{n-2}(z)$ is not bounded. In particular the asymptotic relation does not hold on the spectrum $S(W)$. However, on X_2 one might find a subsequence for which the asymptotic result holds.

COROLLARY. *Suppose that condition* (1.3) *is fulfilled, then as* $n \to \infty$,

$$\text{(i)} \quad \frac{q_{2n}(z)}{q_{2n-1}(z)} \sim \frac{Q(z)+b_1^2}{z-a_1}, \tag{2.4}$$

$$\text{(ii)} \quad \frac{q_{2n+1}(z)}{q_{2n}(z)} \sim \frac{Q(z)+b_2^2}{z-a_2} \tag{2.5}$$

uniformly on compact subsets of $\bar{\mathbb{C}} \backslash (X_1 \cup X_2)$.

Proof. From the recurrence formula (1.1) we easily obtain

$$\frac{q_{2n+1}(z)}{q_{2n-1}(z)} = (z-\alpha_{2n}) \frac{q_{2n}(z)}{q_{2n-1}(z)} - \beta_{2n};$$

and since the left-hand side converges uniformly on compact subsets of $\bar{\mathbb{C}} \backslash (X_1 \cup X_2)$, (2.4) follows immediately by letting $n \to \infty$. The same is true for (2.5). ∎

THEOREM 2. *Under the condition* (1.3) *we have as* $n \to \infty$,

$$\frac{1}{n} \frac{q_n'(z)}{q_n(z)} \sim \frac{z-(a_1+a_2)/2}{\sqrt{[(z-a_1)(z-a_2)-(b_1^2+b_2^2)]^2-4b_1^2 b_2^2}} \tag{2.6}$$

uniformly on every compact subset of $\bar{\mathbb{C}} \backslash (X_1 \cup X_2)$.

Proof. The sequence $q_n(z)/q_{n-2}(z)$ converges uniformly on compact subsets of $\bar{\mathbb{C}} \backslash (X_1 \cup X_2)$ to $Q(z)$, then also the sequence of derivatives $(q_n(z)/q_{n-2}(z))'$ converges uniformly on compact subsets of $\bar{\mathbb{C}} \backslash (X_1 \cup X_2)$ to $Q'(z)$ [17, Theorem 10.28]. Taking derivatives of (2.3) yields

$$\frac{q_n'(z)}{q_n(z)} - \frac{q_{n-2}'(z)}{q_{n-2}(z)} \sim \frac{Q'(z)}{Q(z)}$$

uniformly on compact subsets of $\bar{\mathbb{C}} \backslash (X_1 \cup X_2)$. Now let K be a compact set in $\bar{\mathbb{C}} \backslash (X_1 \cup X_2)$, then

$$\frac{1}{2n} \frac{q_{2n}'(z)}{q_{2n}(z)} = \frac{1}{2n} \sum_{j=N}^{n} \left(\frac{q_{2j}'(z)}{q_{2j}(z)} - \frac{q_{2j-2}'(z)}{q_{2j-2}(z)} \right) + \frac{1}{2n} \frac{q_{2N-2}'(z)}{q_{2N-2}(z)},$$

where N is such that $q_n(x)$ has no zeros for $x \in K$ and $n \geqslant 2N-2$. Then Cesaro's lemma leads to

$$\frac{1}{2n} \frac{q_{2n}'(z)}{q_{2n}(z)} \sim \frac{1}{2} \frac{Q'(z)}{Q(z)}$$

uniformly on every compact subset of $\bar{\mathbb{C}}\backslash(X_1 \cup X_2)$. A similar reasoning holds for odd indexes. Then explicitly calculating $Q'(z)/Q(z)$ gives the desired result. ▌

Observe that both the asymptotic behaviors of the ratios $q_n(z)/q_{n-1}(z)$ and $q_n'(z)/q_n(z)$ only depend on the limits of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, and not on these sequences themselves. So we have invariant asymptotic behavior for these functions.

III. Invariant Quadrature Formulas

In this section some applications of the results of the previous section are given. We will use the concept of *weak convergence* for this purpose [4]. A sequence of distribution functions $F_n(x)$ converges weakly to a distribution function $F(x)$, which is denoted by $F_n(x) \Rightarrow F(x)$, if for every continuous and bounded function $f(x)$

$$\int_{-\infty}^{+\infty} f(x)\, dF_n(x) \to \int_{-\infty}^{+\infty} f(x)\, dF(x). \tag{3.1}$$

A very useful transform in the theory of orthogonal polynomials is the *Stieltjes transform.* If $F(x)$ is a distribution function, i.e., a nondecreasing real function with $F(-\infty)=0$ and $F(\infty)=1$, then its Stieltjes transform is defined as

$$S(F(x); z) = \int_{-\infty}^{+\infty} \frac{dF(x)}{z-x}, \qquad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.2}$$

This function is analytic in both the sets $\{z \mid \operatorname{Im} z > 0\}$ and $\{z \mid \operatorname{Im} z < 0\}$ and it determines the function $F(x)$ uniquely if we normalize $F(x)$ to be right continuous. A way to prove weak convergence is to prove that the Stieltjes transform $S(F_n(x); z)$ converges to the Stieltjes transform $S(F(x); z)$ uniformly on every compact subset of $\mathbb{C}\backslash\mathbb{R}$. This statement is known as the *Grommer–Hamburger theorem* [3, Appendix].

In this paper we will need the Stieltjes transform of two distribution functions: let $|\gamma| \leqslant \alpha < \beta$ and define

$$F(x; \alpha, \beta) = \frac{1}{\pi} \int_{-\infty}^{x} \frac{|t|}{\sqrt{\beta^2 - t^2}\,\sqrt{t^2 - \alpha^2}}\, I_B(t)\, dt, \tag{3.3}$$

$$G(x; \alpha, \beta, \gamma) = \frac{2}{\pi} \frac{[(\alpha^2-\gamma^2)^{1/2} + (\beta^2-\gamma^2)^{1/2}]^2}{(\beta^2-\alpha^2)^2} \times \int_{-\infty}^{x} \frac{\sqrt{\beta^2-t^2}\,\sqrt{t^2-\alpha^2}}{|t-\gamma|}\, I_B(t)\, dt, \tag{3.4}$$

where $I_B(t)$ is the indicator function of the set $B=[-\beta, -\alpha]\cup[\alpha,\beta]$. A straightforward but tedious calculation yields

LEMMA. *Let $z\in\mathbb{C}\backslash[-\beta,-\alpha]\cup[\alpha,\beta]$, then*

$$\text{(i)}\quad S(F(x;\alpha,\beta);z)=\frac{z}{\sqrt{z^2-\alpha^2}\sqrt{z^2-\beta^2}}, \tag{3.5}$$

$$\text{(ii)}\quad S(G(x;\alpha,\beta,\gamma);z)$$
$$=\frac{2(z+\gamma)}{z^2-\gamma^2-[(\alpha^2-\gamma^2)(\beta^2-\gamma^2)]^{1/2}+\sqrt{z^2-\alpha^2}\sqrt{z^2-\beta^2}}. \tag{3.6}$$

The roots $\sqrt{z^2-\alpha^2}$ and $\sqrt{z^2-\beta^2}$ are chosen such that $z^2/\sqrt{z^2-\alpha^2}\sqrt{z^2-\beta^2}$ is analytic in $\mathbb{C}\backslash([-\beta,-\alpha]\cup[\alpha,\beta])$ and tends to 1 as z tends to infinity.

From now on we denote $a_{(1)}=\min(a_1,a_2)$, $a_{(2)}=\max(a_1,a_2)$ and the same for $b_{(1)}$ and $b_{(2)}$. The Christoffel numbers $\{\lambda_{j,n}\}$ of the orthogonal polynomials $\{q_n(x)\}$ are defined to be the unique numbers such that the Gauss–Jacobi mechanical quadrature

$$\int p(x)\,dW(x)=\sum_{j=1}^{n}\lambda_{j,n}\,p(x_{j,n})$$

holds for every polynomial $p(x)$ of degree at most $2n-1$ ([18], p 47).

THEOREM 3. *Suppose that condition* (1.3) *is fulfilled. Let $\{p_n(x)\}$ be othonormal polynomials that satisfy* (1.3) *and $\{\lambda_{j,n}\}$ its Christoffel numbers, then for every continuous function $f(x)$*

$$\text{(i)}\quad \sum_{j=1}^{2n}\lambda_{j,2n}\,p_{2n-1}^2(x_{j,2n})\,f(x_{j,2n}) \tag{3.7}$$
$$\to\frac{b_{(1)}^2}{b_1^2}\int_{-\infty}^{+\infty}f(x)\,dG\left(x-\frac{a_1+a_2}{2};\alpha,\beta,\gamma\right)+\frac{b_1^2-b_{(1)}^2}{b_1^2}f(a_2),$$

$$\text{(ii)}\quad \sum_{j=1}^{2n+1}\lambda_{j,2n+1}\,p_{2n}^2(x_{j,2n+1})\,f(x_{j,2n+1}) \tag{3.8}$$
$$\to\frac{b_{(1)}^2}{b_2^2}\int_{-\infty}^{+\infty}f(x)\,dG\left(x-\frac{a_1+a_2}{2};\alpha,\beta,-\gamma\right)+\frac{b_2^2-b_{(1)}^2}{b_2^2}f(a_1),$$

where we have put

$$\alpha^2 = \left(\frac{a_1 - a_2}{2}\right)^2 + (b_1 - b_2)^2,$$
$$\beta^2 = \left(\frac{a_1 - a_2}{2}\right)^2 + (b_1 + b_2)^2, \tag{3.9}$$
$$\gamma = \frac{a_1 - a_2}{2},$$

and the function $G(x; \alpha, \beta, \gamma)$ *is as in* (3.4).

Proof. It is very elementary to write

$$\frac{q_{n-1}(z)}{q_n(z)} = \sum_{j=1}^{n} \frac{a_{j,n}}{z - x_{j,n}}, \tag{3.10}$$

where the numbers $a_{j,n}$ are equal to $q_{n-1}(x_{j,n})/q_n'(x_{j,n})$. It is well known [18, p. 48] that

$$\lambda_{j,n} = \frac{\gamma_n}{\gamma_{n-1}} \frac{1}{p_{n-1}(x_{j,n})\, p_n'(x_{j,n})};$$

hence $a_{j,n} = \lambda_{j,n}\ p_{n-1}^2(x_{j,n})$. Define a discrete distribution function that makes jumps at the zeros $\{x_{j,n}\}$ by

$$G_n(x) = \sum_{j=1}^{n} a_{j,n} U(x - x_{j,n}), \tag{3.11}$$

where

$$U(x) = 1, \qquad x \geqslant 0,$$
$$= 0, \qquad x < 0.$$

Note that every $a_{j,n}$ is positive and that

$$\sum_{j=1}^{n} a_{j,n} = \sum_{j=1}^{n} \lambda_{j,n} p_{n-1}^2(x_{j,n})$$
$$= \int_{-\infty}^{+\infty} p_{n-1}^2(x)\, dW(x) = 1,$$

which follows from the Gauss–Jacobi mechanical quadrature. (Note that we used this already in the proof of Theorem 1.) The Stieltjes transform of $G_n(x)$ is by (3.10),

$$S(G_n(x); z) = \sum_{j=1}^{n} \frac{a_{j,n}}{z - x_{j,n}} = \frac{q_{n-1}(z)}{q_n(z)},$$

and if we use the corollary to Theorem 1 we have

$$S(G_{2n}(x); z) \to \frac{z-a_1}{b_1^2+Q(z)},$$

$$S(G_{2n+1}(x); z) \to \frac{z-a_2}{b_2^2+Q(z)},$$

uniformly on every compact subset of $\mathbb{C}\backslash\mathbb{R}$ as $n \to \infty$. Some tedious calculations enable us to write

$$\frac{z-a_1}{b_1^2+Q(z)} = \frac{(b_1+b_2-|b_1-b_2|)^2}{4b_1^2}(2z+a_1-a_2)$$
$$\times\left\{\left(z-\frac{a_1+a_2}{2}\right)^2-\left(\frac{a_1-a_2}{2}\right)^2-|b_1^2-b_2^2|\right.$$
$$\left.+\sqrt{(z-a_1)(z-a_2)-(b_1-b_2)^2}\sqrt{(z-a_1)(z-a_2)-(b_1+b_2)^2}\right\}^{-1}$$
$$+\frac{|b_1^2-b_2^2|-(b_2^2-b_1^2)}{2b_1^2}\frac{1}{z-a_2}.$$

Use (3.6) to identify the first term as

$$\frac{b_{(1)}^2}{b_1^2}S\left(G\left(x-\frac{a_1+a_2}{2};\alpha,\beta,\gamma\right);z\right)$$

with α, β,and γ as in (3.9). The second term is easily seen to be the Stieltjes transform of

$$\frac{b_1^2-b_{(1)}^2}{b_1^2}U(x-a_2),$$

and since weak convergence is equivalent to (3.1) we have the result in (i). The result in (ii) follows by interchanging (a_1, b_1) by (a_2, b_2). ∎

Next we will give a result on the zero distribution of orthogonal polynomials that satisfy (1.3). This result has also been obtained by Geronimus [11, p. 76], but by other methods.

THEOREM 4. *If condition* (1.3) *is fulfilled, then for every continuous function* $f(x)$

$$\frac{1}{n}\sum_{j=1}^{n} f(x_{j,n}) \to \int_{-\infty}^{+\infty} f(x)\, dF\left(x-\frac{a_1+a_2}{2};\alpha,\beta\right), \tag{3.12}$$

where α *and* β *are as in* (3.9) *and* $F(x;\alpha,\beta)$ *is as in* (3.3).

Proof. As in the previous proof we use a decomposition into partial fractions to obtain

$$\frac{q_n'(z)}{q_n(z)} = \sum_{j=1}^{n} \frac{1}{z - x_{j,n}}.$$

Now we define the distribution functions

$$F_n(x) = \frac{1}{n} \sum_{j=1}^{n} U(x - x_{j,n}); \tag{3.13}$$

hence $nF_n(x)$ equals the number of zeros of $q_n(x)$ less than or equal to x. The Stieltjes transform is

$$S(F_n(x); z) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{z - x_{j,n}} = \frac{1}{n} \frac{q_n'(z)}{q_n(z)},$$

so that from (2.6) we may conclude that

$$S(F_n(x); z) \to \frac{z - (a_1 + a_2)/2}{\sqrt{[(z - a_1)(z - a_2) - (b_1^2 + b_2^2)]^2 - 4b_1^2 b_2^2}}$$

uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$. Then use (3.5) to identify this limit as the Stieltjes transform of $F(x - (a_1 + a_2)/2;\ \alpha, \beta)$. Hence $F_n(x) \Rightarrow F(x - (a_1 + a_2)/2;\ \alpha, \beta)$ and from (3.1) we obtain the desired result. ∎

This last theorem gives the asymptotic zero distribution of the polynomials $\{q_n(x)\}$: first, we mention that the theorem is also true for every bounded measurable function $f(x)$ which is continuous almost everywhere [4, Theorem 5.2.iii], i.e., for Riemann integrable functions. If we take $f(x)$ to be the indicator function of the interval $[p, q]$, then

$$\frac{N(n; p, q)}{n} \to \int_p^q dF\left(x - \frac{a_1 + a_2}{2};\ \alpha, \beta\right),$$

where $N(n; p, q)$ is the number of zeros of $q_n(x)$ in the interval $[p, q]$. It turns out that as $n \to \infty$ the number of zeros of $q_n(x)$ outside $[-\beta, -\alpha] \cup [\alpha, \beta]$ is of order $o(n)$ (where $o(n)/n$ tends to zero if $n \to \infty$), and in these intervals the zeros are more dense near the endpoints $\pm\alpha$ and $\pm\beta$.

IV. Special Cases

We will now consider some special cases of the previous theorems. The most important case is when $a_1 = a_2 = a$ and $b_1 = b_2 = b > 0$. The functions F and G in Theorems 3 and 4 then have the form

$$F(x-a; 0, 2b) = \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{\sqrt{4b^2 - (t-a)^2}} I_{[a-2b, a+2b]}(t)\, dt,$$

$$G(x-a; 0, 2b, 0) = \frac{1}{2b^2\pi} \int_{-\infty}^{x} \sqrt{4b^2 - (t-a)^2}\, I_{[a-2b, a+2b]}(t)\, dt.$$

Theorems 1–4 then coincide with results of Nevai [13]. Theorem 4 is a slight generalization of Maki's result [12]. Another important case is when $b_1 = b_2 = b > 0$. For this case Chihara [6, 7] proved that the zeros are dense in the set

$$\left[\frac{a_1 + a_2}{2} - \left\{\left(\frac{a_1 - a_2}{2}\right)^2 + 4b^2\right\}^{1/2}, a_{(1)}\right]$$
$$\cup \left[a_{(2)}, \frac{a_1 + a_2}{2} + \left\{\left(\frac{a_1 - a_2}{2}\right)^2 + 4b^2\right\}^{1/2}\right]$$

and this is just the set on which the limiting zero distribution function $F(x - (a_1 + a_2)/2;\ \frac{1}{2}\ |a_1 - a_2|,\ \{(a_1 - a_2)/2 + 4b^2\}^{1/2})$ is concentrated. The result of Theorem 4, however, is stronger than Chihara's result since it indicates how dense the zeros are distributed in the set mentioned higher. The case $a_1 = a_2 = a$ and the general case are new, except for the zero distribution which was also obtained by Geronimus [11].

Special attention should be paid to the case where b_1 or b_2 is equal to zero. We will formulate these in a theorem.

Theorem 5. *Suppose that* (1.3) *is satisfied and that* $b_{(1)} = 0$. *Then for every continuous function* $f(x)$

$$\text{(i)} \quad \sum_{j=1}^{2n} \lambda_{j,2n} p_{2n-1}^2(x_{j,2n}) f(x_{j,2n})$$

$$\to \begin{cases} f(a_2) & \text{if } b_2 = 0 \\ \dfrac{1}{2\beta}\left\{\left(\dfrac{a_2 - a_1}{2} + \beta\right) f\left(\dfrac{a_1 + a_2}{2} + \beta\right) \right. & \\ \qquad \left. + \left(\dfrac{a_1 - a_2}{2} + \beta\right) f\left(\dfrac{a_1 + a_2}{2} - \beta\right)\right\} & \text{if } b_1 = 0, \end{cases}$$

(ii) $\sum_{j=1}^{2n+1} \lambda_{j,2n+1} p_{2n}^2(x_{j,2n+1}) f(x_{j,2n+1})$

$$\to \begin{cases} f(a_1) & \text{if } b_1 = 0 \\ \dfrac{1}{2\beta}\left\{\left(\dfrac{a_1 - a_2}{2} + \beta\right) f\left(\dfrac{a_1 + a_2}{2} + \beta\right)\right. \\ \qquad \left. + \left(\dfrac{a_2 - a_1}{2} + \beta\right) f\left(\dfrac{a_1 + a_2}{2} - \beta\right)\right\} & \text{if } b_2 = 0, \end{cases}$$

(iii) $\dfrac{1}{n}\sum_{j=1}^{n} f(x_{j,n}) \to \dfrac{1}{2}\left\{f\left(\dfrac{a_1 + a_2}{2} + \beta\right) + f\left(\dfrac{a_1 + a_2}{2} - \beta\right)\right\}$,

where $\beta^2 = ((a_1 - a_2)/2)^2 + b_{(2)}^2$.

Proof. Equations (2.4) and (2.5) become for this case,

$$\frac{q_{2n}(z)}{q_{2n-1}(z)} \to \begin{cases} z - a_2 - \dfrac{b_2^2}{z - a_1} & \text{if } b_1 = 0 \\ z - a_2 & \text{if } b_2 = 0, \end{cases}$$

$$\frac{q_{2n+1}(z)}{q_{2n}(z)} \to \begin{cases} z - a_1 - \dfrac{b_1^2}{z - a_2} & \text{if } b_2 = 0 \\ z - a_1 & \text{if } b_1 = 0, \end{cases}$$

while (2.6) turns out to be

$$\frac{1}{n}\frac{q_n'(z)}{q_n(z)} \to \left(z - \frac{a_1 + a_2}{2}\right) \Big/ ((z - a_1)(z - a_2) - b_{(2)}^2).$$

Hence the Stieltjes transforms of the functions $G_n(x)$ in (3.11) satisfy uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$

$$S(G_{2n}(x); z) \to \begin{cases} \dfrac{1}{z - a_2} & \text{if } b_2 = 0 \\ \dfrac{1}{2\beta}\left\{\dfrac{(a_2 - a_1)/2 + \beta}{z - (a_1 + a_2)/2 - \beta} + \dfrac{(a_1 - a_2)/2 + \beta}{z - (a_1 + a_2)/2 + \beta}\right\} & \text{if } b_1 = 0 \end{cases}$$

and interchanging (a_1, b_1) and (a_2, b_2) gives a similar result for $S(G_{2n+1}(x); z)$. From these asymptotics (i) and (ii) are immediate. The Stieltjes transforms of the functions $F_n(x)$ in (3.13) have the behaviour

$$S(F_n(x); z) \to \frac{1}{2}\left\{\frac{1}{z - (a_1 + a_2)/2 - \beta} + \frac{1}{z - (a_1 + a_2)/2 + \beta}\right\}$$

uniformly on every compact subset of $\mathbb{C}\backslash\mathbb{R}$, from which (iii) follows. ∎

Hence, whenever one of b_1 or b_2 is zero the distribution functions $G_n(x)$ and $F_n(x)$ converge weakly to a distribution function that makes at most two jumps. This means that for large n most of the zeros will be concentrated around these points where the limit distribution function makes a jump.

V. Examples

In this section we will give some sequences of orthogonal polynomials to which the above results apply. We use the terminology in Chihara [7]. Examples 1–5 also follow from Nevai's work but are given here to get a better understanding of the results in the previous sections.

Example 1. *Jacobi polynomials* satisfy (1.1) with

$$\alpha_n = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$

$$\beta_n = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}$$

with $\alpha > -1$ and $\beta > -1$. It is easy to see that $\alpha_n \to 0$ and $\beta_n \to \frac{1}{4}$, hence we obtain uniformly on every compact subset of $\bar{\mathbb{C}} \setminus [-1, 1]$

$$\begin{aligned} \frac{q_n(z)}{q_{n-1}(z)} &\sim \frac{z+\sqrt{z^2-1}}{2}, \\ \frac{1}{n}\frac{q_n'(z)}{q_n(z)} &\sim \frac{1}{\sqrt{z^2-1}}, \end{aligned} \tag{5.1}$$

and for every continuous function $f(x)$

$$\begin{aligned} \sum_{j=1}^{n} \lambda_{j,n} p_{n-1}^2(x_{j,n}) f(x_{j,n}) &\to \frac{2}{\pi}\int_{-1}^{1} f(x)\sqrt{1-x^2}\,dx \\ \frac{1}{n}\sum_{j=1}^{n} f(x_{j,n}) &\to \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx. \end{aligned} \tag{5.2}$$

These results are well known.

Example 2. *Pollaczek polynomials* have

$$\alpha_n = \frac{-b}{n+\lambda+a+c},$$

$$\beta_n = \frac{(n+c)(n+2\lambda+c-1)}{4(n+\lambda+a+c-1)(n+\lambda+a+c)},$$

where $a \geqslant |b|$ together with $2\lambda + c > 0$ and $c \geqslant 0$ or $2\lambda + c \geqslant 1$ and $c > -1$. Since $\alpha_n \to 0$ and $\beta_n \to \frac{1}{4}$ as $n \to \infty$ we have the same asymptotics as in (5.1) and (5.2).

EXAMPLE 3. *Modified Lommel polynomials* have the coefficients

$$\alpha_n = 0,$$
$$\beta_n = \frac{1}{4(n+\nu)(n+\nu-1)},$$

with $\nu > 0$, so that $\beta_n \to 0$. Hence we have a degenerate limit and we have, uniformly on every compact subset of $\bar{\mathbb{C}} \setminus (\{0\} \cup \{\text{mass points of } W\})$

$$\begin{aligned} &\frac{q_n(z)}{q_{n-1}(z)} \sim z, \\ &\frac{1}{n} \frac{q_n'(z)}{q_n(z)} \sim \frac{1}{z}, \end{aligned} \tag{5.3}$$

and for every continuous function $f(x)$

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_{j,n} p_{n-1}^2(x_{j,n}) f(x_{j,n}) \to f(0), \\ &\frac{1}{n} \sum_{j=1}^{n} f(x_{j,n}) \to f(0). \end{aligned} \tag{5.4}$$

That these limits are degenerate at 0 can perhaps be understood by the fact that the modified Lommel polynomials are orthogonal with respect to a discrete distribution function $W(x)$ that makes jumps at the points

$$\{j_{\nu-1,k}^{-1};\, k = 0, \pm 1, \pm 2, \ldots\},$$

where $\cdots < j_{\nu,-1} < j_{\nu,0} < 0 < j_{\nu,1} < \cdots$, denote the zeros of the Bessel function $J_\nu(x)$, and this set has 0 as its only limit point.

EXAMPLE 4. *Tricomi–Carlitz polynomials*, where

$$\alpha_n = 0,$$
$$\beta_n = \frac{n}{(n+\alpha)(n+\alpha-1)},$$

with $\alpha > 0$. Again $\beta_n \to 0$ and we have the same asymptotics as in (5.3) and (5.4). Note that also in this case the polynomials are orthogonal with respect to a discrete distribution function, and the jumps now are at $\{\pm 1/\sqrt{k+\alpha};\ k=0, 1, 2,...\}$. This set also has 0 as its only limit point.

EXAMPLE 5. The *q-polynomials of Al-Salam and Carlitz* satisfy (1.1) with

$$\alpha_n = (1+a)\, q^n,$$
$$\beta_n = -aq^{n-1}(1-q^n),$$

where $a<0$ and $0<q<1$. Therefore both α_n and β_n converge to 0 as $n \to \infty$ and Eqs. (5.3) and (5.4) are valid. Once again these polynomials are orthogonal with respect to a discrete distribution function and the jumps occur at the points $\{q^k, aq^k;\ k=0, 1, 2,...\}$ which again has 0 as its only limit point.

Now we will also give an example where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ have two limit points:

EXAMPLE 6. Consider the sequence of polynomials $\{q_n(x)\}$ that satisfy (1.1) with

$$\alpha_{2n} = a_1; \qquad \beta_{2n} = b_1^2;$$
$$\alpha_{2n+1} = a_2; \qquad \beta_{2n+1} = b_2^2. \tag{5.5}$$

Obviously they satisfy condition (1.3) so that Theorems 1–4 are valid here. Moreover, we can explicitely give the distribution function $W(x)$ with respect to whom these polynomials are orthogonal. Nevai pointed out to me that the distribution function was also given by Geronimus [10].

THEOREM 6. *The polynomials $\{q_n(x)\}$ with the coefficients as in* (5.5) *are orthogonal with respect to*

$$W(x) = \frac{b_{(1)}^2}{b_1^2}\, G\left(x - \frac{a_1+a_2}{2}; \alpha, \beta, -\gamma\right) + \frac{b_1^2 - b_{(1)}^2}{b_1^2}\, U(x-a_1)$$

where α, β and γ are as in (3.9) *and $G(x;\alpha,\beta,\gamma)$ is as in* (3.4).

Proof. Consider the polynomials $\{q_n^*(x)\}$ with coefficients as in (5.5) but with b_1 and b_2 interchanged. From (1.1) we easily obtain

$$\frac{q_{2n+1}^*(z)}{q_{2n}^*(z)} = (z-a_1) - b_2^2\,\frac{q_{2n-1}^*(z)}{q_{2n}^*(z)},$$
$$\frac{q_{2n+2}^*(z)}{q_{2n+1}^*(z)} = (z-a_2) - b_1^2\,\frac{q_{2n}^*(z)}{q_{2n+1}^*(z)}.$$

From the corollary to Theorem 1 we know that the limits of the functions in these equations exist:

$$\frac{q^*_{2n+1}(z)}{q^*_{2n}(z)} \to Q_1(z),$$

$$\frac{q^*_{2n+2}(z)}{q^*_{2n+1}(z)} \to Q_2(z).$$

Let $n \to \infty$ in those equations to obtain

$$Q_1(z) = (z - a_1) - b_2^2 \frac{1}{Q_2(z)},$$

$$Q_2(z) = (z - a_2) - b_1^2 \frac{1}{Q_1(z)},$$

so that, when continuously combining these equations, we find the continued fraction.

$$\frac{1}{Q_1(z)} = \cfrac{1}{z - a_1 - \cfrac{b_2^2}{z - a_2 - \cfrac{b_1^2}{z - a_1 - \cfrac{b_2^2}{z - a_2 - \cfrac{b_1^2}{\ddots}}}}}$$

This is a so-called *Jacobi fraction*, and from the theory of continued fractions [20, 7] it follows that

$$\frac{1}{Q_1(z)} = S(W(x); z),$$

where $W(x)$ is the distribution function with respect to whom the polynomials $\{q_n(x)\}$ with coefficients as in (5.5) are orthogonal. As in the proof of Theorem 3 we have

$$\frac{1}{Q_1(z)} = \frac{b_{(1)}^2}{b_1^2} S\left(G\left(x - \frac{a_1 + a_2}{2}; \alpha, \beta, -\gamma\right); z\right)$$
$$+ \frac{b_1^2 - b_{(1)}^2}{b_1^2} S(U(x - a_1); z),$$

from which the theorem follows. ∎

Special cases of these polynomials where already studied by Chihara. When $a_1 = a_2 = a$ and $b_1 = b_2 = b$, the polynomials $q_n(x)$ are equal to $U_n((x-a)/2b)$, where $U_n(x)$ is the Chebyshev polynomial of the second kind of degree n. For $a_1 = -c$; $a_2 = c$ and $\beta_n = \frac{1}{4}$ Chihara [5] obtained the weight function

$$w(x) = \left(\frac{|x+c|}{|x-c|}\right)^{1/2} (1 + c^2 - x^2)^{1/2}, \qquad c^2 \leqslant x^2 \leqslant 1 + c^2,$$
$$= 0, \qquad \text{elsewhere},$$

and for $a_1 = a_2 = 0$ and $b_1 = b$; $b_2 = a$ the weight function was found to be

$$w(x) = \frac{1}{|x|}\left[\left(1 - \left(\frac{x^2 - a^2 - b^2}{2ab}\right)^2\right]^{1/2}, \qquad x \in [-(a+b), -|a-b|] \cup [|a-b|, a+b],$$
$$= 0, \qquad \text{elsewhere},$$

and in addition there is a jump at 0 [7, p. 91].

References

1. N. I. Akhiezer, Uber eine Eigenschaft der "elliptischen" Polynome, *Comm. Soc. Math. Kharkov* **4** (1934), 3–8.
2. N. I. Akhiezer and M. G. Krein, Some questions in the theory of moments, *in* Transl. Math. Monographs, Vol. 2", Amer. Math. Soc., Providence, R. I., 1962.
3. L. Arnold, Deterministic version of Wigner's semicircle law for the distribution of matrix eigenvalues, *Linear Algebra Appl.* **13** (1976), 185–199.
4. P. Billingsley, "Convergence of Probability Measures," Wiley, New York, 1968.
5. T. S. Chihara, On kernel polynomials and related systems, *Boll. Un. Mat. Ital.* (3) **19** (1964), 451–459.
6. T. S. Chihara, Orthogonal polynomials whose zeros are dense in intervals, *J. Math. Anal. Appl.* **24** (1968), 362–371.
7. T. S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon & Breach, New York, 1978.
8. P. Erdös and P. Turan, On interpolation III, *Ann. of Math.* **41** (1940), 510–555.
9. G. Freud, "Orthogonal Polynomials," Pergamon, Oxford, 1971.
10. Ya. L. Geronimus, On some finite-difference equations and the corresponding system of orthogonal polynomials, *in* "*Mem. Math. Section Faculty of Math. & Phys. Kharkov State Univ. and Kharkov Math. Soc.*, Vol. 25," pp. 87–100, 1957. [in Russian]
11. Ya. L. Geronimus, "Orthogonal Polynomials (Appendix to the Russian translation of [18])," Fizmatgiz, Moscow, 1961; English translation *in* "Amer. Math. Soc. Transl., Vol. 108," pp. 37–130, 1977.
12. D. Maki, On determining regular behavior from the recurrence formula for orthogonal polynomials, *Pacific J. Math.* **91** (1980), 173–178.
13. P. G. Nevai, "Orthogonal Polynomials," Mem. Amer. Math. Soc. Vol. 213, Amer. Math. Soc., Providence, R. I., 1979.

14. P. G. NEVAI, Distribution of zeros of orthogonal polynomials, *Trans. Amer. Math. Soc.* **249** (1979), 341–361.
15. P. G. NEVAI AND J. S. DEHESA, On asymptotic average properties of zeros of orthogonal polynomials, *SIAM J. Math. Anal.* **10** (1979), 1184–1192.
16. H. POINCARÉ, Sur les équations lineaires aux differentielles ordinaires et aux differences finies, *Amer. J. Math.* **7** (1885), 203–258.
17. W. RUDIN, "Real and complex Analysis," 2nd ed., Tata McGraw–Hill, New Delhi, 1978.
18. G. SZEGÖ, "Orthogonal polynomials," Amer. Math. Soc. Coll. Publ., Vol. 23, Amer. Math. Soc., Providence, R. I., 1967.
19. J. L. ULLMAN, On the regular behaviour of orthogonal polynomials, *Proc. London Math. Soc.* (3) **24** (1972), 119–148.
20. H. S. WALL, "Analytic Theory of Continued Fractions," Chelsea, Bronx, N.Y. 1973.